THEORY OF NONLINEAR WAVES IN A PLASMA

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Stationary nonlinear waves propagating in a cold rarefied plasma composed of electrons and two types of ions are considered. The structure of isolated waves and shock waves is found. In recent years an intensive study has been made of finite-amplitude waves and collisionless shock waves in a rarefied plasma, in connection with laboratory experiments [1] and astrophysical applications (the problem of the interaction of the "solar wind" with the Earth's magnetosphere [2]). When allowance is made for dispersion effects associated with the departure of the dispersion law $\omega = \omega(k)$ from the linear, and for the compensating nonlinear twisting of the wave profile, we are able to obtain the profile of stationary nonlinear waves of finite amplitude, and when allowance is made for damping we can also obtain the structure of a collisionless shock wave [3]. Such waves have been studied fairly fully for the case of a two-component plasma. The present paper examines stationary nonlinear waves propagating across a magnetic field in a cold rarefied quasi-neutral plasma composed of electrons and two types of ions.

The nature of the dispersion law $\omega = \omega(k)$ for small-amplitude waves in the three-component plasma under consideration is illustrated in Fig. 1 (see also [4]); here and in what follows

$$\omega_{1,2} = \frac{eH_0}{m_{1,2}c}, \qquad \omega^{(0)} = \left(\frac{m_1}{m_2}\alpha_1 + \alpha_2\right)\omega_1, \qquad \alpha_j = \frac{n_{j0}}{n_0}.$$

The index 1 corresponds to the heavier type of ion, n_{j0} is the unperturbed density of the j-th ion type, n_0 is the unperturbed electron density. At low frequencies the phase velocity of small oscillations is

$$\frac{\omega}{k} = \frac{H_0}{\sqrt{4\pi n_0(m_1\alpha_1 + m_2\alpha_2)}} \equiv V_A$$

and decreases as ω approaches

$$\omega^{(\infty)} = \frac{eH_0}{\sqrt{m_1 m_2}c}\left(\frac{m_1\alpha_1 + m_2\alpha_2}{m_1\alpha_2 + m_2\alpha_1}\right)^{1/2}.$$

For comparatively small nonlinear wave velocities the lower branch of the dispersion curve is basically the "operative" one, and the characteristic dimensions of the compression waves which may exist for such a dispersion law are equal in order of magnitude to $\delta \sim V_A/\omega^{(\infty)}$, which gives

$$\delta \sim \frac{c\sqrt{m_2}}{\sqrt{4\pi n_0 c^2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{1/2}$$

in the interesting case of a small admixture of a light component or a large difference in ion masses $m_1\alpha_1 \gg m_2\alpha_2$.

As the wave velocity increases, the upper branch begins to play a part, and the nonlinear wave profile changes, as will be clear from what follows. For frequencies $\omega \gg \omega^{(\infty)}$ the upper branch of the dispersion curve has the asymptote

$$\frac{\omega}{k} = V_A\left[\left(\alpha_1 + \frac{m_2\alpha_2}{m_1}\right)\left(\alpha_1 + \frac{m_1\alpha_2}{m_2}\right)\right]^{1/2}.$$

In the region of frequencies large compared with ω_1, ω_2, the departure of this branch from linear behavior becomes quite clear for the hybrid frequency

$$\omega_h = \frac{eH_0}{\sqrt{m_i m_e}c}$$

but this region has been thoroughly studied (dispersion effects associated with electron inertia) and is not considered here.

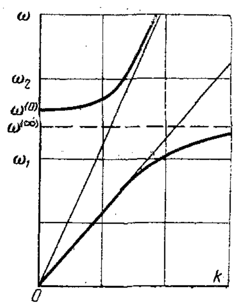

Fig. 1

We now pass to the study of nonlinear stationary waves. In a coordinate system moving with the wave velocity U the appropriate equations may be written in the form

$$m_j v_{jx}\frac{dv_{jx}}{dx} = eE_x + \frac{e}{c}v_{jy}H, \qquad m_j v_{jx}\frac{dv_{jy}}{dx} = \frac{e}{c}(UH_0 - v_{jx}H),$$
$$\frac{dH}{dx} = -\frac{4\pi e n_0}{c}\left(U\sum_{j=1}^{2}\alpha_j\frac{v_{jy}}{v_{jx}} + \frac{cE_x}{H_0}\right). \qquad (1)$$

Here the index j = 1, 2 determines the type of ion, the x axis is in the direction of motion of the plasma ahead of the wave, the z axis coincides with the direction of the magnetic field, H_0 is the unperturbed magnetic field.

The electron velocity is determined from the drift approximation, since we are interested in the region of frequencies $\omega \ll \omega_h$. By means of fairly simple transformations we obtain the equation

$$A^2 v\frac{d}{dx}\left\{\frac{v}{vh - \alpha_1 M}\left(1 + \frac{m_1}{m_1\alpha_1 + m_2\alpha_2}v\frac{dv}{dh}\right)\frac{dh}{dx}\right\} = M - vh, \quad (2)$$

where

$$\frac{1}{2}\frac{A^2v^2}{(vh-\alpha_1 M)^2}\left(1+\frac{m_1}{m_1\alpha_1+m_2\alpha_2}v\frac{dv}{dh}\right)^2\left(\frac{dh}{dx}\right)^2 =$$
$$= -\left(1+\frac{m_1\alpha_2}{m_1\alpha_1+m_2\alpha_2}\right)(h-1)-\frac{1}{2}\frac{m_1}{m_1\alpha_1+m_2\alpha_2}(v^2-M^2)-$$
$$-\frac{2m_2\alpha_2^2M^2}{m_1\alpha_1+m_2\alpha_2}\left\{\frac{1}{(h_2-h_1)(h_3-h_1)}\ln\left|\frac{h-h_1}{1-h_1}\right|+\right.$$
$$+\frac{1}{(h_1-h_2)(h_3-h_2)}\ln\left|\frac{h-h_2}{1-h_2}\right|+$$
$$\left.+\frac{1}{(h_2-h_3)(h_1-h_3)}\ln\left|\frac{h-h_3}{1-h_3}\right|\right\}, \qquad (3)$$

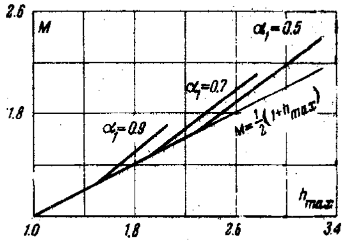

Fig. 2

If the condition $m_1\alpha_1 \gg m_2\alpha_2$ is fulfilled, then the velocity of the heavy component (3) assumes the simple form

$$v = M\left(1+\frac{1-h^2}{2M^2}\right)$$

and equation (2) may be integrated once

$$A^2 = \frac{c^2m_1m_2\alpha_2}{4\pi n_0(m_1\alpha_1+m_2\alpha_2)e^2}, \qquad M=\frac{U}{V_A}, \qquad h=\frac{H}{H_0},$$

$$v=\frac{v_{1x}}{V_A}=\frac{M}{2m_1\alpha_1 h}\left\{m_1\alpha_1^2-m_2\alpha_2^2+(m_1\alpha_1+m_2\alpha_2)\times\right.$$
$$\times h\left(1+\frac{1-h^2}{2M^2}\right)+((m_1\alpha_1^2-m_2\alpha_2^2)^2-$$
$$-2(m_1\alpha_1+m_2\alpha_2)(m_1\alpha_1^2+m_2\alpha_2^2)h\left(1+\frac{1-h^2}{2M^2}\right)+$$
$$\left.+(m_1\alpha_1+m_2\alpha_2)^2h^2\left(1+\frac{1-h^2}{2M^2}\right)^2\right)^{1/2}\right\}. \qquad (4)$$

Here h_1, h_2, h_3 are the roots of the equation $h^3 - (2M^2+1)h + 2M^2\alpha_1 = 0$. Choice of the constant of integration corresponds to an isolated wave. Equation (4) allows us to establish a connection between the velocity of an isolated wave and the maximum magnetic field strength in the wave. This function is given in Fig. 2 for different values of the relative concentrations of different types of ions. For small amplitudes of the magnetic field the velocity of an isolated wave is equal to

$$M = {}^1/_2\,(1+h_{\max}). \qquad (5)$$

As h_{max} increases, the wave velocity increases more rapidly than is given by formula (5). This departure from linearity sets in all the more rapidly as the relative concentration of the light component decreases.

In the region where the velocity M depends linearly on the amplitude of the magnetic field h_{max} the profile of an isolated wave takes the familiar form of a symmetrical hump. In the case of small amplitudes $h - 1 \ll 1$ we may obtain an analytical expression for the magnetic field in an isolated wave

$$h = 1 + b_{\max}\,\mathrm{sech}^2\left(\frac{\sqrt{b_{\max}}}{2A}x\right), \qquad b_{\max} = h_{\max} - 1.$$

For large wave velocites when the relation $M = M(h_{max})$ is nonlinear, the form of an isolated wave changes radically, since the upper branch of the dispersion curve begins to play a part. A typical isolated wave profile in a three-component plasma at large velocities, obtain by numerical solution of the system of equations (1), is represented in Fig. 3. As the concentration of the light component decreases, the linear dimension of the isolated wave also decreases, which is in agreement with estimates from the linear theory given above; in addition to this, the size of the "hollow" in the center of the wave decrases. We note that the solution referred to exists only for wave velocities less than a certain critical value, which depends on the relative concentrations of the light and heavy components. The singular point of system of equations (1) corresponding to the unperturbed state of the plasma in front of the wave is a saddle point (the integral curve starts out from this point), if the condition

$$1 < M < \left[\left(\frac{m_1}{m_2}\alpha_2+\alpha_1\right)\left(\alpha_1+\frac{m_2}{m_1}\alpha_2\right)\right]^{1/2}$$

is fulfilled.

The lower limit of the wave velocity is, of course, equal to the velocity of sound, and the upper limit is equal to the phase velocity of small oscillations for frequencies which are large compared with ω_1, ω_2. As the wave velocity approaches this upper limit, the light component is "swept out" of the wave, and dispersion effects compensating the nonlinear twisting appear at higher frequencies

$$\omega \sim \frac{eH_0}{\sqrt{m_i m_e}\,c}$$

and depend upon the electron inertia. The critical value of the wave velocity, and consequently the maximal magnetic field in the wave, decreases as the relative concentration of the light component decreases. Thus for small concentrations of the lighter type of ion the amplitudes of isolated waves in the frequency range

$$\omega \ll \frac{eH_0}{\sqrt{m_i m_e}\,c}$$

will be small.

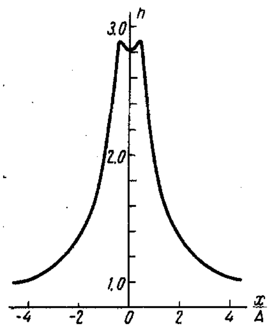

Fig. 3

We shall estimate the energy of ions in the wave. The energy of heavy ions in the direction of wave motion ("longitudinal" energy) is equal in order of magnitude to

$$\mathscr{E}_x^{(1)} \sim m_1 n_1 v_{1x}^2 \sim m_1 n_1 \left(\frac{eE_x}{m_1\omega}\right)^2.$$

The "transverse" energy of light ions is equal in order of magnitude to

$$\mathscr{E}_y^{(2)} \sim m_2 n_2 v_{2y}^2 \sim m_2 n_2 \left(\frac{cE_x}{H}\right)^2.$$

Taking into account the fact that

$$\omega \sim \omega^{(\infty)} \approx \sqrt{\omega_1 \omega_2 \alpha_2 / \alpha_1}$$

we obtain $\mathscr{E}_y^{(2)} / \mathscr{E}_x^{(1)} \sim 1$, and the energies for a single frequency are estimated as follows:

$$\mathscr{E}_y^{(2)} / \mathscr{E}_x^{(1)} \sim n_1 / n_2$$

Whence it follows that when the isolated wave considered above propagates in a three-component plasma an accelerating mechanism occurs which accelerates the light ions in a direction perpendicular to the direction of wave motion.

If the friction between the plasma components is introduced into the initial equations, we obtain a shock wave with an oscillatory structure having a steep leading edge. The profile of such a wave is depicted qualitatively in Fig. 4.

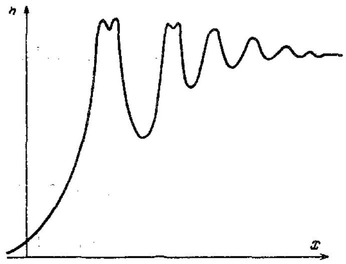

Fig. 4

REFERENCES

1. R. Kh. Kurtmullaev, Yu. E. Nesterikhin, V. I. Pil'skii, and R. Z. Sagdeev, "Mechanism of plasma heating by collisionless shock waves," International Conference, Kalem, England, 1965.

2. V. D. Pletnev, G. A. Skuridin, and L. S. Chesalin, Kosmicheskie isspedovaniya, vol. 3, no. 3, p. 408, 1965.

3. R. Z. Sagdeev, "Collective processes and shock waves in a rarefied plasma," collection: Problems of Plasma [in Russian], Atomizdat, no. 4, 1964.

4. V. L. Yakimenko, Zh. tekh. fiz., vol. 32, no. 2, p. 168, 1962.

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